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On the Sraffa-Leontief Model

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Abstract

In this paper we consider the matrix forms of the Sraffa-Leontief income distribution model $px = (1 + r)pAx + w^*p(I - A)x$ introduced by Steenge (1995, 1997). We will explore the equivalence between these matrix forms and the set of simpler models, including the Sraffian condition of linear relations between the rate of profits r and wage rate w^* . Further, we will evaluate the condition that the price vector p and the commodities vector x are the left-hand and the right-hand eigenvectors of the matrix A of direct inputs and that these vectors are the fixed points of the Sraffian standard commodities-standard prices matrix. We will then explore links between the Sraffa-Leontief system and the multiplier product matrix (MPM) for the matrix A to consider new insights generated through visualization with the help an artificial economic landscape. Furthermore, the connections between MPM and the Sraffian standard commodities-standard prices matrix and their minimal information properties are proven.

Keywords: Sraffa-Leontief income distribution model; Multiplier product matrix; Sraffian standard commodities-standard prices matrix; Minimum information properties.

JEL classification: C67; D51; B41

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1. INTRODUCTION

The contributions of Sraffa (1960) to the understanding of economic structure have been significantly advanced in recent years by the interpretative assessments of Steenge (1995, 1997). In this paper, these interpretations are complemented with some additional modifications that attempt to simplify the presentation of the Sraffa-Leontief system. In the next section, the standard Sraffian model is presented and some of the initial modifications are outlined. In Section 3, the concept of the multiplier product matrix (MPM) is introduced but in a modified form; instead of considering the Leontief inverse matrix, the MPM methodology is applied to the matrix of direct coefficients to afford a direct link with the Sraffa system. In Section 4, the minimum information properties of the MPM and Sraffian matrix are presented and are shown to be directly related in Sections 5 and 6. Section 7 provides an empirical example from an input-output table for the Chicago metropolitan region. We conclude with some brief summary remarks and suggestions for further extensions.

2. THE SRAFFIAN MODEL

2.1 Sraffian Prices Decomposition Matrix Primary Model and Commodities Decomposition Matrix Dual Model

The simplest and most obvious way to construct the Sraffian income distribution model is as follows (*cf.* Pasinetti 1977; see also Kurz and Salvadori 1995 for a more extensive treatment): the input-output model is defined in the usual manner.

$$(1) \quad x = Ax + f,$$

where x is a vector of gross output, A the matrix of input coefficients, and f is a vector of final demand.

Introducing the vector of prices p we obtain:

$$(2) \quad px = pAx + pf,$$

where it is conventionally assumed that the price of final demand is defined by the amount of labor imbedded in gross output:

$$(3) \quad wx = pf.$$

(We assume further that $w = I$, i.e., $lx = pf$.) Income distribution theory requires (see Pasinetti 1977, p.72-73, (V.3.1a)):

$$(4) \quad p = (I + r)pA + w^*l,$$

where r is the uniform rate of profits and w^* is the wage rate and $0 \leq w^* \leq 1$.

From (2), (3), and (4) the Sraffa-Leontief income distribution model may be obtained:

$$(5) \quad px = (1+r)pAx + w^* p(I-A)x.$$

This scalar model can be derived from the price decomposition matrix primal model:

$$(6) \quad p = (1+r)pA + w^* p(I-A),$$

or from the commodities decomposition matrix dual model:

$$(7) \quad x = (1+r)Ax + w^*(I-A)x,$$

by post-multiplication of (6) on x and pre-multiplication of (7) on p .

2.2 Sraffian Matrix and Decomposition of the Matrix A of Direct Inputs

Consider a non-negative matrix, A , of direct input coefficients. (In the following, it is not necessary to adopt the usual assumption of decomposability and primitivity of matrix A .) From the theory of non-negative matrices (see, for example, Horn and Johnson 1985, p. 503), it follows that there are non-negative eigenvectors (left hand and right hand) p^* and x^* of A corresponding to a non-negative eigenvalue μ .

$$(8) \quad p^* A = \mu p^*, \quad Ax^* = \mu x^*$$

such that μ is a simple eigenvalue.

It is well known that if the sum of the elements of each column of the matrix of direct inputs is less than 1 and, of course, greater than 0, then the maximum eigenvalue μ is in the interval $0 \leq \mu < 1$. This follows from the well-known inequality for the spectral radius $\rho(A)$ of eigenvalues of positive matrix A (see, for example, Horn and Johnson 1985, p. 346):

$$(9) \quad \min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}.$$

The simplicity of μ means that the eigenspaces of all left-hand and right-hand eigenvectors corresponding to μ are one-dimensional spaces, i.e., each left-hand eigenvector is proportional to p^* , and each right hand eigenvector is proportional to x^* .

Let us assume (following Steenge 1995, p. 57) that

$$(10) \quad p^* x^* = 1$$

and consider the matrix $S = x^*p^*$. This matrix will assume a major role in the subsequent analysis, and it will be referred to as the *Sraffian matrix* with the vector x^* , the vector of standard commodities, and the vector p^* , the vector of standard prices. Note that this definition of standard commodities is different from the conventional Sraffa definition: Sraffa labeled as standard commodities the vector of final demand f^* generating the gross output x^* :

$$f^* = x^* - Ax^* = (1 - \mu) x^*.$$

For standardization, we will assume a constant value for the Sraffian standard commodity

$$(11) \quad pf^* = p^* f = p^* f^* = (1 - \mu).$$

Condition (10) means that these vectors are fixed points of the transformation S :

$$(12) \quad p^*S = p^*, \quad Sx^* = x^*.$$

The conditions (8) imply that

$$(13) \quad AS = SA = \mu S.$$

Consider further $C = I - S$. Obviously

$$(14) \quad \begin{aligned} S^2 &= S, \quad C^2 = C, \quad CS = SC = 0, \\ AS &= SA = \mu S, \quad AC = CA \end{aligned}$$

i.e.,

$$(15) \quad A = A(S + C) = \mu S + CA.$$

Goodwin (1983) and Steenge (1995) used, in their considerations, the complicated fine structure (Perronean properties) of the spectrum and spectral decomposition of the matrices with non-negative components. Alternatively, use will be made here only of the decomposition (15), the Perronean eigenvector—corresponding to the larger eigenvalue – and the one-dimensional eigenspace that splits from the eigenvalues structure. This splitting essentially will simplify the proofs.

2.3 Sraffa's Linear Wage-Profit Trade-Off

If in the Sraffian model (5), an arbitrary vector of prices p and gross output equal to x^* are chosen, then substituting gross output x by the eigenvector x^* in the Sraffa-Leontief model (5), using (8) and dividing by px^* one obtains:

$$(16) \quad 1 = (1 + r)\mu + w^*(1 - \mu).$$

This relationship implies the Sraffa linear relation between the rate of profits r and the wage rate w^* (see Sraffa 1960, p. 22; see also Pasinetti 1977, p. 115):

$$(17) \quad r = (1 - w^*) \frac{1 - \mu}{\mu}.$$

Obviously, the maximal rate of profits corresponding to the case $w^* = 0$ is equal to:

$$(18) \quad r_{\max} = \frac{1 - \mu}{\mu}.$$

The same situation will occur when the choice is made of an arbitrary gross output x and the vector of prices p^* .

2.4 Equivalence Theorems for the Primary Sraffa-Leontief Matrix Model

The linear wage-profit trade-off (equations 17 - 18) implies the following statements.

Theorem P1. *If (α) in the Primary Sraffa-Leontief matrix model (6) $w^* \neq 1$, then (β) in this model is equivalent to*

$$(19) \quad \begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ \mu p = pA \end{cases}$$

Proof: $(\alpha) \Rightarrow (\beta)$.

The model (6) can be rewritten in the following form:

$$(20) \quad (1 - w^*)p = (1 + r - w^*)pA$$

and, as shown above in Section 2.3, implies (17 - 18). The conditions (17 - 18) mean that:

$$(21) \quad 1 - w^* = \mu(1 + r - w^*),$$

and this implies, after substitution in (20), that:

$$(22) \quad \mu p = pA$$

$$(\beta) \Rightarrow (\alpha).$$

Contrarily, if the conditions (19) are true, then multiplying (22) by $1 + r - w^*$ and using (21) one obtains (20); this concludes the proof of equivalence.

2.5 The Steenge Equivalence Condition

The following proposition is the reformulation and clarification of the considerations of Steenge (1995, pp. 63-66). Steenge proved that in Propositions P1 and D1, both conditions in (β) are equivalent in the Sraffa model. In his proof, he used the fine structure of the all eigenvalues and eigenvectors of positive matrices.

Proposition P1. *If (α) in the Primary Sraffa-Leontief matrix model (7) $w^* \neq 1$, then (β) this model is equivalent to*

$$(23) \quad \begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ p = pS \end{cases}.$$

Proof: $(\alpha) \Rightarrow (\beta)$

If condition (22) is true then

$$\mu pC = \mu p(I - S) = \mu p - \mu pS = pA - \mu pS = p(A - \mu S) = pAC.$$

This condition means that the vector pC is a left-hand eigenvector for A ; and because the eigenspace corresponding to the simple eigenvalue μ is one-dimensional, then the vector pC is proportional to p^* , i.e., there is a number λ such that $pC = \lambda p^*$. Using (9), we obtain $pC = \lambda p^*S$. Therefore $pC = pC^2 = (\lambda p^*S)C = 0$. Thus,

$$(24) \quad p = pS + pC = pS$$

$$(\beta) \Rightarrow (\alpha).$$

Conversely, if $p = pS$, then $pC = 0$, and $pCA = pAC = 0$. Therefore

$$pA = p(\mu S + AC) = \mu pS = \mu p.$$

Thus, (23) is equivalent to (19).

2.6 Equivalence Theorems for the Dual Sraffa-Leontief Matrix Model

Analogously, the following statement can be proven for the Dual Sraffa-Leontief model (7).

Theorem D1. *If (α) in the Primary Sraffa-Leontief matrix model (7) $w^* \neq 1$, then (β) in this model is equivalent to*

$$(25) \quad \begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ \mu x = Ax \end{cases}$$

Proposition D1 (Steenge, 1995). *If (α) in the Dual Sraffa-Leontief matrix model (7) $w^* \neq 1$, then (β) this model is equivalent to*

$$(26) \quad \begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ x = Sx \end{cases}.$$

3. THE MULTIPLIER PRODUCT MATRIX (MPM)

3.1 The Definition of MPM

In this section, a connection between the Sraffa standard commodity system and the multiplier product matrix will be revealed. The definition of the direct inputs multiplier product matrix (MPM) is as follows: let $A = [a_{ij}]$ be a matrix of direct inputs in the input-output system, and let $m_{\bullet j}$ and $m_i \bullet$ be the column and row sums of this matrix. Following Chenery and Watanabe (1958) these are defined as:

$$(27) \quad m_{\bullet j} = \sum_{i=1}^n a_{ij}, \quad m_i \bullet = \sum_{j=1}^n a_{ij}.$$

Let V be the global intensity of the matrix A :

$$(28) \quad V = V(A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}.$$

Then, the input-output multiplier product matrix (MPM) is defined as:

$$(29) \quad M = M(A) = \frac{1}{V} \|m_{i \bullet} m_{\bullet j}\| = \frac{1}{V} \begin{pmatrix} m_{1\bullet} \\ m_{2\bullet} \\ \vdots \\ m_{n\bullet} \end{pmatrix} (m_{\bullet 1} \quad m_{\bullet 2} \quad \cdots \quad m_{\bullet n}) = [m_{ij}]$$

Introducing the vectors of column and row sums:

$$(30) \quad M_r(A) = \begin{pmatrix} m_{1\bullet} \\ m_{2\bullet} \\ \vdots \\ m_{n\bullet} \end{pmatrix}; \quad M_c(A) = (m_{\bullet 1} \quad m_{\bullet 2} \quad \cdots \quad m_{\bullet n})$$

one obtains the following expression of MPM matrix:

$$(31) \quad M(A) = \frac{1}{V} M_r(A) M_c(A).$$

The properties of the MPM will now be considered in the context of the following issues: (i) the hierarchy of backward and forward direct inputs linkages and their economic landscape associated with the cross-structure of the MPM; and (ii) the minimum information properties of the MPM.

3.2 Economic Cross-Structure Landscapes of MPM and the Rank-Size Hierarchies of Input Backward and Forward Linkages

In this subsection, the main notions and results of the Rasmussen-Hirshman key sector analysis of backward and forward linkages for the direct and indirect inputs of Leontief inverse will be transferred to the case of direct inputs. To this end, the transformation in the classical theory of key sectors (*cf.* Sonis, Hewings, and Guo 2000; Sonis and Hewings 1999) involves replacement of the Leontief inverse by the consideration of the matrix of direct inputs, A . Following this analogy and the ideas of Rasmussen (1956), two types of indices will be defined, drawing on entries in the matrix A of direct inputs:

1. Power of dispersion of direct inputs for the backward linkages, $DIBL_j$, as follows.

$$(32) \quad \begin{aligned} DIBL_j &= \frac{1}{n} \sum_{i=1}^n a_{ij} \bigg/ \frac{1}{n^2} \sum_{i,j=1}^n a_{ij} = \\ &= \frac{1}{n} m_{\bullet j} \bigg/ \frac{1}{n^2} V = m_{\bullet j} \bigg/ \frac{1}{n} V \end{aligned}$$

and

2. The indices of the sensitivity of dispersion of direct inputs for forward linkages, $DIFL_i$, as follows.

$$(33) \quad \begin{aligned} DIFL_i &= \frac{1}{n} \sum_{j=1}^n a_{ij} \bigg/ \frac{1}{n^2} \sum_{i,j=1}^n a_{ij} = \\ &= \frac{1}{n} m_{i\bullet} \bigg/ \frac{1}{n^2} V = m_{i\bullet} \bigg/ \frac{1}{n} V \end{aligned}$$

A direct inputs key sector, K , is usually defined as one in which both indices are greater than 1.

The definitions of backward and forward linkages provided by (32) and (33) imply that the rank-size hierarchies (rank-size ordering) of these indices coincide with the rank-size hierarchies of the column and row sums. In this connection, it is important to underline that the column and row sums for MPM are the same as those for the matrix of direct inputs A :

$$(34) \quad \begin{aligned} \sum_{j=1}^n m_{ij} &= \frac{1}{V} \sum_{j=1}^n m_{i\bullet} m_{\bullet j} = m_{i\bullet} \\ \sum_{i=1}^n m_{ij} &= \frac{1}{V} \sum_{i=1}^n m_{i\bullet} m_{\bullet j} = m_{\bullet j} \end{aligned}$$

Thus, the structure of the MPM is essentially connected with the properties of sectoral direct inputs backward and forward linkages.

The structure of the matrix M can be ascertained in the following fashion: consider the largest column sum $m_{\bullet j}$ and the largest row sum $m_{i\bullet}$ of the matrix A . Further, the

element $m_{i_0 j_0} = \frac{1}{V} m_{i_0 \bullet} m_{\bullet j_0}$ is located in the place (i_0, j_0) of the matrix M . Moreover, all

rows of the matrix M are proportional to the i_0^{th} row, and the elements of this row are larger than the corresponding elements of all other rows. The same property applies to the j_0^{th} column of the same matrix. Hence, the element located in (i_0, j_0) defines the center of the largest *cross* within the matrix M . If this cross is excluded from M , then the second largest cross can be identified and so on. Thus, the matrix M contains the rank-size sequence of crosses. One can reorganize the locations of rows and columns of M in such a way that the centers of the corresponding crosses appear on the main diagonal. In this fashion, the matrix will be reorganized in such a way that a descending *economic landscape* will be apparent (see Figure 1).

This rearrangement also reveals the descending rank-size hierarchies of the indices for direct forward and backward linkages. Inspection of that part of the landscape with indices > 1 (the criterion for specification of direct inputs key sectors) will enable the identification of the key sectors (see Figure 2). However, it is important to stress that the construction of the economic landscape for different regions or for the same region at different points in time would create the possibility for the establishment of taxonomy of these economies. Moreover, the superposition of the hierarchy of one region on the landscape of another region provides a clear visual representation of the similarities and differences in the linkage structure of these regions.

It is important to stress, as will be shown in Section 6, that the Sraffian standard commodities-standard prices matrix S coincides with the multiplier product matrix $M(S)$. Hence, the Sraffian matrix has the same cross-structure defined by rank-size hierarchies of components of vectors of standard commodities and standard prices (see Figure 3).

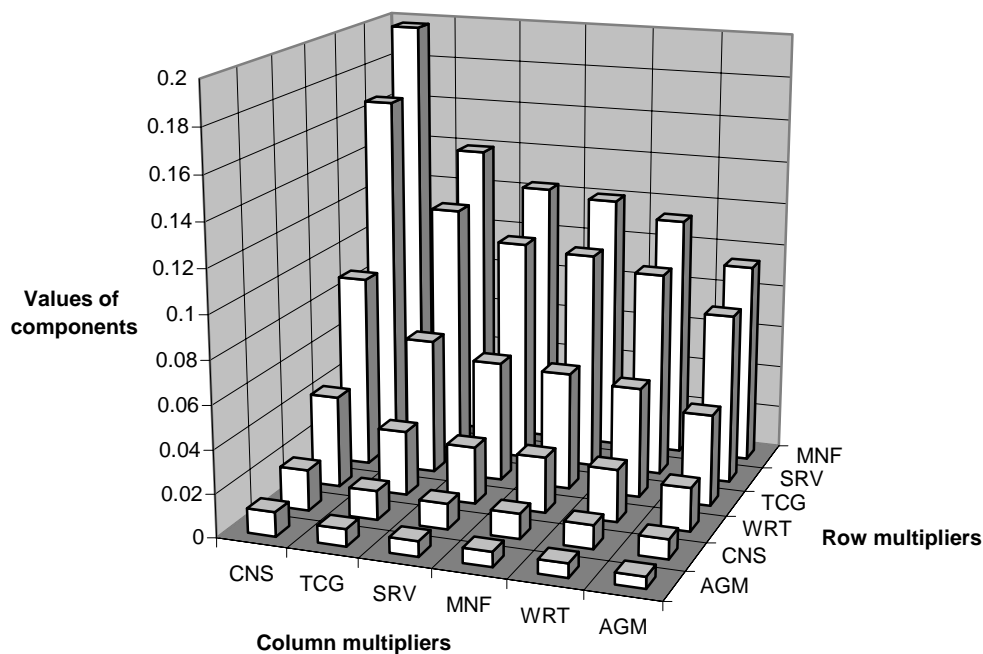


FIGURE 1. Economic Landscape for Chicago 1987 Input-Output Table

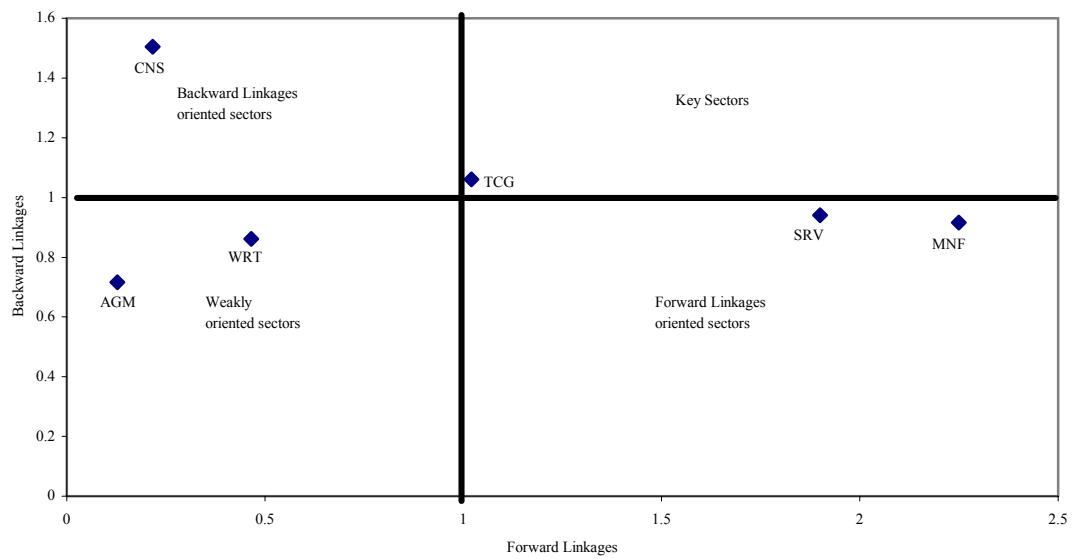


FIGURE 2. Direct Inputs Backward and Forward Linkages, Chicago, 1987

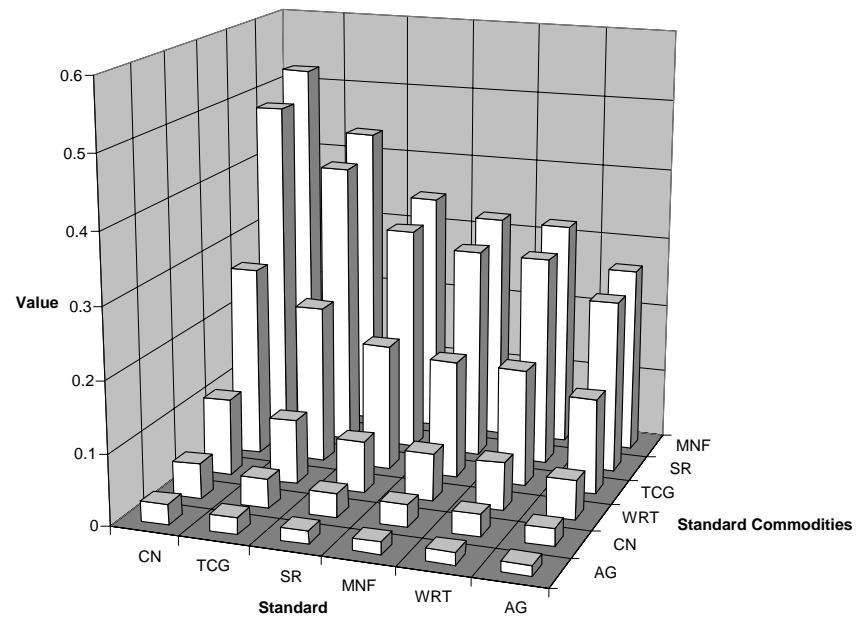


FIGURE 3. Economic Landscape of Sraffian Matrix

4. MINIMUM INFORMATION PROPERTIES OF THE MPM AND S

4.1 Definition of Information of the Positive Matrix

Consider all positive matrices, $\Psi = [\Psi_{ij}]$ with the property that the row and column multipliers are equal to those of the matrix A :

$$(35) \quad \sum_j \Psi_{ij} = m_{i\bullet}, \quad \sum_i \Psi_{ij} = m_{\bullet j}.$$

Obviously, $\sum_{i,j} \Psi_{ij} = V$. We can convert each positive matrix Ψ into the two-dimensional probabilistic distribution matrix, $P = [p(i, j)]$ with the components:

$$(36) \quad p(i, j) = \Psi_{ij} / V.$$

Therefore, we can attribute to each positive matrix Ψ the Shannon information (INF):

$$(37) \quad INF\Psi = INFP = \sum_{i,j} p(i, j) \ln p(i, j) = \sum_{i,j} \frac{\Psi_{ij}}{V} \ln \frac{\Psi_{ij}}{V}.$$

4.2 Minimum Information of MPM and S

Recall the well-known Shannon information inequality (Shannon and Weaver 1964, p. 51):

$$(38) \quad \sum_{i,j} p(i, j) \ln p(i, j) \geq \sum_{i,j} p(i, j) \ln \sum_j p(i, j) + \sum_{i,j} p(i, j) \ln \sum_i p(i, j).$$

This implies that each positive matrix ψ satisfying the condition (39) may be shown as:

$$\begin{aligned} (39) \quad INF\Psi &= \sum_{i,j} \frac{\Psi_{ij}}{V} \ln \frac{\Psi_{ij}}{V} \geq \sum_{ij} \frac{\Psi_{ij}}{V} \ln \sum_j \frac{\Psi_{ij}}{V} + \sum_{ij} \frac{\Psi_{ij}}{V} \ln \sum_i \frac{\Psi_{ij}}{V} = \\ &= \sum_{ij} \frac{\Psi_{ij}}{V} \ln \frac{m_{i\bullet}}{V} + \sum_{ij} \frac{\Psi_{ij}}{V} \ln \frac{m_{\bullet j}}{V} = \sum_i \left(\sum_j \frac{\Psi_{ij}}{V} \right) \ln \frac{m_{i\bullet}}{V} + \sum_j \left(\sum_i \frac{\Psi_{ij}}{V} \right) \ln \frac{m_{\bullet j}}{V} = \\ &= \sum_i \frac{m_{i\bullet}}{V} \ln \frac{m_{i\bullet}}{V} + \sum_j \frac{m_{\bullet j}}{V} \ln \frac{m_{\bullet j}}{V} = \sum_i \frac{m_{i\bullet}}{V^2} \left(\sum_j m_{\bullet j} \right) \ln \frac{m_{i\bullet}}{V} + \sum_j \frac{m_{\bullet j}}{V^2} \left(\sum_i m_{i\bullet} \right) \ln \frac{m_{\bullet j}}{V} = \\ &= \sum_{ij} \frac{m_{i\bullet} \cdot m_{\bullet j}}{V^2} \left(\ln \frac{m_{i\bullet}}{V} + \ln \frac{m_{\bullet j}}{V} \right) = \sum_{ij} \frac{m_{i\bullet} \cdot m_{\bullet j}}{V^2} \ln \frac{m_{i\bullet} \cdot m_{\bullet j}}{V^2} = INFM. \end{aligned}$$

Then:

$$(40) \quad INF\Psi \geq INFM$$

and the multiplier product matrix M has a minimal information property (Sonis 1968).

The matrix M may be considered to represent the most homogeneous distribution of the components of the column and row sums of the matrix A . A further perspective may be offered; in the case of equal column and row sums, the economic landscape will be a flat, horizontal plane.

The MPM depends on the column and row sums only and thus represents only the aggregate characteristics of the direct interactions of each sector with the rest of the economy. Thus, the MPM does not take into account the specifics of the pair-wise sectoral interactions between direct inputs; the MPM can be considered as an aggregate representation of some sector equalization tendency in the economic interaction between sectors. Of course the same property of minimal information hold for the Sraffian matrix.

5. PROPERTIES OF COLUMN AND ROW MULTIPLIERS OF THE PRODUCT OF POSITIVE MATRICES

In the following, this important property of the multipliers of the product of two positive matrices will be used. Consider the product $A = A_1 A_2 = [a_{ij}]$ of two matrices $A_1 = [a_{ij}^1]$, $A_2 = [a_{ij}^2]$. Let

$$(41) \quad \begin{aligned} m_{\bullet j} &= \sum_{i=1}^n a_{ij}, & m_i \bullet &= \sum_{j=1}^n a_{ij} \\ m_{\bullet j}^1 &= \sum_{i=1}^n a_{ij}^1, & m_i^1 \bullet &= \sum_{j=1}^n a_{ij}^1 \\ m_{\bullet j}^2 &= \sum_{i=1}^n a_{ij}^2, & m_i^2 \bullet &= \sum_{j=1}^n a_{ij}^2 \end{aligned}$$

be the column and row multipliers of these matrices. Let $V = \sum_{i,j} a_{ij}$ be the global intensity of the matrix A . Further, specify the following vectors of column and row multipliers.

$$(42) \quad \begin{aligned} M_c(A) &= [m_{\bullet 1} m_{\bullet 2} \dots m_{\bullet n}]; M_c(A_1) = [m_{\bullet 1}^1 m_{\bullet 2}^1 \dots m_{\bullet n}^1]; M_c(A_2) = [m_{\bullet 1}^2 m_{\bullet 2}^2 \dots m_{\bullet n}^2] \\ M_r(A) &= \begin{bmatrix} m_{1 \bullet} \\ m_{2 \bullet} \\ \vdots \\ m_{n \bullet} \end{bmatrix}; M_r(A_1) = \begin{bmatrix} m_{1 \bullet}^1 \\ m_{2 \bullet}^1 \\ \vdots \\ m_{n \bullet}^1 \end{bmatrix}; M_r(A_2) = \begin{bmatrix} m_{1 \bullet}^2 \\ m_{2 \bullet}^2 \\ \vdots \\ m_{n \bullet}^2 \end{bmatrix} \end{aligned}$$

The following formulae can be checked by direct calculations of the components of corresponding vectors and matrices.

$$(43) \quad \begin{aligned} M_c(A) &= M_c(A_1)A_2; \\ M_r(A) &= A_1M_r(A_2); \\ V(A) &= M_c(A_1)M_r(A_2) \end{aligned}$$

6. INTERCONNECTIONS BETWEEN MPM AND S

It is obvious that the standardization condition of vectors x^* and p^* means that:

$$(44) \quad M_c(S) = Xp^*; \quad M_r(S) = Px^*$$

where X and P are the sums of the components of vectors of standard commodities x^* and standard prices p^*

$$(45) \quad X = \sum_i x_i^*; \quad P = \sum_i p_i^*.$$

Thus, the multipliers of the Sraffian matrix S are:

$$(46) \quad M_c(S) = Xp^*; \quad M_r(S) = Px^*$$

and $V(A) = PX$. Hence, the multiplier product matrix of the Sraffian matrix coincides with the Sraffian matrix itself:

$$(47) \quad M(S) = S.$$

Using this property and applying formulae (43) to the condition $AS = SA = \mu S$ one obtains:

$$(48) \quad M_c(A)S = \mu Xp^*; \quad SM_r(A) = \mu Px^*.$$

This implies that

$$(49) \quad M_c(A)x^* = \mu X; \quad p^* M_r(A) = \mu P$$

or

$$(50) \quad p^* M(A)x^* = \frac{\mu^2 PX}{V} = \mu^2.$$

Further, using the well-known property of the row multipliers:

$$(51) \quad M_c(A) = e - v_a$$

where $e = (1, 1, \dots, 1)$ and vector v_a is the vector of value added, the conditions (43) and (45) imply:

$$\mu p^* = M_r(A)S = eS - v_a S = Xp^* - v_a S$$

or

$$(52) \quad v_a S = (X - \mu)p^*.$$

The standardization (11) helps find the exact expression for vector of prices p and vector of commodities x that are the solutions of the Sraffian models: conditions $\mu p = pA$ and $\mu x = Ax$ means that p and x are the eigenvectors of A corresponding to the eigenvalue μ . Therefore they are proportional to p^* and x^* :

$$p = \alpha p^*, \quad x = \beta x^*.$$

Introducing them into the standardization condition (11), one obtains $\alpha = \beta = 1 - \mu$, so the solutions for the Sraffa models are:

$$(53) \quad p = (1 - \mu)p^*; \quad x = (1 - \mu)x^*.$$

Substituting (53) into (5) and (6), the decomposition of prices and commodities in Sraffa-Leontief models into three parts may be obtained, namely, an intermediate inputs part, an interest part, and a wage part:

$$(54) \quad \begin{aligned} p &= (1 - \mu)p^* = (1 - \mu)p^* A + r(1 - \mu)p^* A + w^*(1 - \mu)p^*(I - A); \\ x &= (1 - \mu)x^* = (1 - \mu)Ax^* + r(1 - \mu)Ax^* + w^*(1 - \mu)(I - A)x^* \end{aligned}$$

7. EXAMPLE

7.1 Multiplier Product Matrix for the Chicago Economy, 1987

As an example, let us consider the 6×6 Chicago 1987 input-output table, aggregated from a 36-sector version. This table was derived from the Chicago Region Econometric Input-Output Model (CREIM), details on which can be found in Israilevich et al. (1997). Table 1 presents the sector definition for the Chicago table, and Table 2 presents the aggregated table itself together with the column and row multipliers, the direct inputs backward and forward indices, and their rank-size hierarchies.

Table 2 represents the matrix of direct inputs A from the Primary and Dual Sraffa-Leontief models (6) and (7). The column and row multipliers of the matrix A define the Multiplier Product matrix, calculated with the help of the formula (31) (see Table 3).

Rearranging the columns and rows of this matrix according the rank-size hierarchies of column and row sums, Table 4 will be revealed; this is the numerical presentation of the economic landscape visualized graphically in Figure 1. The spatial presentation of the hierarchies of backward and forward linkages of the matrix A is presented in the space of backward and forward linkages in Figure 2. If a time series of matrices is available, such a representation will be useful for the description of the temporal dynamics of these linkages.

7.2 Sraffian Matrix for the Chicago Economy in 1987

The column sums from Table 2, defined with the help of (10), reveal the interval that includes the maximal Perronean eigenvalue: $0.2501 < \mu < 0.5261$. The actual value of this maximal eigenvalue is $\mu = 0.3367$, so the maximal rate of profits in Chicago 1987 economy is

$$r_{\max} = \frac{1 - \mu}{\mu} = 1.97 .$$

The left-hand and right-hand eigenvectors corresponding to this eigenvalue are: standard prices eigenvector,

$$p^* = [0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704], \quad P = 6.1968 ;$$

TABLE 1

Chicago 1987 Input-Output Table Sectors Definitions

Sectors		Description	SIC codes
Aggregate	Original		
<i>AGM</i>	1	Livestock and Other Agricultural Products	01,02
	2	Forestry and Fishery; Agricultural Services	07-09
	3	Mining	10-14
<i>CNS</i>	4	Construction	15-17
<i>MNF</i>	5	Food and Kindred Products	20
	6	Tobacco Manufactures	21
	7	Textiles and Apparel	22-23
	8	Lumber and wood Products	24
	9	Furniture and Fixtures	25
	10	Paper and Allied Products	26
	11	Printing and Publishing	27
	12	Chemicals and Allied Products	28
	13	Petroleum Refining and Related Industries	29
	14	Rubber and Miscellaneous Plastics Products	30
	15	Leather and Leather Products	31
	16	Stone, Clay, Glass and Concrete Products	32
	17	Primary Metal Industries	33
	18	Fabricated Metal products	34
	19	Machinery, Except Electrical	35
	20	Electrical and Electronic Machinery	36
	21	Transportation Equipment	37
	22	Scientific Instruments, Photographic and Medical Goods	38
	23	Miscellaneous Manufacturing Industries	39
<i>TCG</i>	24	Transportation and Warehousing	40-42, 44-47
	25	Communication	48
	26	Electric, Gas and Sanitary Services	49
<i>WRT</i>	27	Wholesale and Retail Trade	50-57, 59
<i>SRV</i>	28	Finance and Insurance	60-64, 67
	29	Real Estate and Rental	65, 66
	30	Hotels, Personal and Business Services	70-73, 76, 81, 89
	31	Eating and Drinking Places	58
	32	Automobile Repair and Services	75
	33	Amusement and Recreation Services	78,79
	34	Health, Educational and Nonprofit Organizations	80, 82-84, 86
	35	Federal Government Enterprises	
	36	State and Local Government Enterprises	

TABLE 2
Chicago 1987 Direct Inputs Table

Sectors	AGM	CNS	MNF	TCG	WRT	SRV	Row Sums	Forward Linkages DIFL	Rank-Size Hierarchy of DIFL
AGM	0.0195	0.0025	0.0101	0.0104	0.0011	0.0012	0.0447	0.1280	VI
CNS	0.0273	0.0008	0.0049	0.0245	0.0054	0.0129	0.0758	0.2168	V
MNF	0.0867	0.3109	0.1707	0.0889	0.0475	0.0816	0.7863	2.2497	I
TCG	0.0245	0.0381	0.0381	0.1542	0.0600	0.0418	0.3567	1.0205	III
WRT	0.0245	0.0631	0.0385	0.0153	0.0133	0.0150	0.1697	0.4856	IV
SRV	0.0676	0.1106	0.0582	0.0776	0.1740	0.1760	0.6639	1.8995	II
Column sums	0.2501	0.5261	0.3204	0.3708	0.3011	0.3287			
Backward Linkages DIBL	0.7156	1.5051	0.9166	1.0608	0.8615	0.9403			
Rank-Size Hierarchy of DIBL	VI	I	IV	II	V	III			

TABLE 3
Multiplier Product Matrix for Chicago 1987
[direct inputs input-output table]

Sectors	AGM	CNS	MNF	TCG	WRT	SRV	Row sums	Rank-Size Hierarchy of Row Sums
AGM	0.00534	0.01124	0.00684	0.00792	0.00643	0.00702	0.0448	VI
CNS	0.00904	0.01902	0.01158	0.01340	0.01088	0.01188	0.0758	V
MNF	0.09377	0.19725	0.12013	0.13902	0.11289	0.12324	0.7863	I
TCG	0.04254	0.08948	0.05450	0.06307	0.05121	0.05591	0.3567	III
WRT	0.02024	0.04257	0.02593	0.03000	0.02436	0.02660	0.1697	IV
SRV	0.07917	0.16655	0.10143	0.11738	0.09532	0.10406	0.6639	II
Column Sums	0.25010	0.52610	0.32040	0.37080	0.30110	0.32870		
Rank-Size Hierarchy of Column Sums	VI	I	IV	II	V	III		

TABLE 4
Economic Landscape for Chicago 1987 Multiplier Product Matrix.

Sectors	CNS	TCG	SRV	MNF	WRT	AGM	Row Sums	Rank-Size Hierarchy of Row Sums
MNF	0.19725	0.13902	0.12324	0.12013	0.11289	0.09377	0.7863	I
SRV	0.16655	0.11738	0.10406	0.10143	0.09532	0.07917	0.6639	II
TCG	0.08948	0.06307	0.05591	0.05450	0.05121	0.04254	0.3567	III
WRT	0.04257	0.03000	0.02660	0.02593	0.02436	0.02024	0.1697	IV
CNS	0.01902	0.01340	0.01188	0.01158	0.01088	0.00904	0.0758	V
AGM	0.01124	0.00792	0.00702	0.00684	0.00643	0.00534	0.0448	VI
Column sums	0.52610	0.37080	0.32870	0.32040	0.30110	0.25010		
Rank-Size Hierarchy of Column Sums	I	II	III	IV	V	VI		

and standard commodities eigenvector,

$$x^* = \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}, X = 1, {}^* p^* x^* = 1 \quad \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}, X = 1, p^* x^* = 1$$

The Sraffian matrix of standard commodities, standard prices for Chicago 1987 economy are presented in Table 5. The corresponding economic landscape for this matrix is shown in Figure 3.

These tables will help us to calculate the decompositions (54) of prices and commodities in Sraffa-Leontief models into three parts: an intermediate inputs part, an interest part, and a wage part (*cf.* Steenge 1997, pp. 244-247). This is shown in the Appendix.

8. CONCLUSIONS AND FURTHER EXPLORATIONS

This paper has revealed an important connection between the Sraffa-Leontief system and some new interpretations afforded by the multiplier product matrix. The properties of the latter matrix offer the potential for comparative analysis across time (for a single economy) or across economies at one point in time. Recently, some further considerations have explored an expansion of the Sraffa-Leontief models by considering a closed system of the Miyazawa-type in which profits and wages are distributed and their impacts on the economy are traced (see Miyazawa 1976; Kimura, Sonis, and Hewings 2002; Sonis and Hewings 2001). In this sense, the work of Trigg (1999), examining a link between Keynes, Morishima, and Miyazawa, provides further motivation for the potentially new and innovative insights that can be gained by exploring connections between modeling systems.

TABLE 5

Sraffian Matrix for Chicago 1987 Direct Inputs Input-Output Table

Sectors	AGM	CNS	MNF	TCG	WRT	SRV	Standard Commodities x^*	Row Multipliers P_{x^*}	Rank-Size Hierarchy of x^*
AGM	0.014204	0.028104	0.017376	0.023405	0.017253	0.018632	0.0192	0.118979	VI
CNS	0.025277	0.049922	0.030861	0.041568	0.030642	0.033091	0.0341	0.211311	V
MNF	0.263961	0.522355	0.322904	0.434939	0.320620	0.346239	0.3568	2.211018	I
TCG	0.136423	0.270108	0.166973	0.224906	0.165792	0.179039	0.1875	1.143310	III
WRT	0.055189	0.109214	0.067513	0.090937	0.067036	0.072392	0.0746	0.462281	IV
SRV	0.244726	0.484291	0.299374	0.403245	0.297257	0.321008	0.3308	2.049901	II
Column Multipliers Xp^*	0.739800	0.464000	0.905000	1.219000	0.898600	0.970400			
Standard Prices p^*		1.464000		1.219000					
Rank-Size Hierarchy of p^*	0.739800		0.905000		0.898600	0.970400			
	VI	I	IV	II	V	III			

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APPENDIX

Decomposition of the Sraffian matrix

$$\begin{aligned}
 &0.6633[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704] = \\
 &0.6633[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704] \begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} + \\
 &+r0.6633[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704] \begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} + \\
 &+w*0.6633[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704] \begin{bmatrix} 0.9805 & -0.0025 & -0.0101 & -0.0104 & -0.0011 & -0.0012 \\ -0.0273 & 0.9992 & -0.0049 & -0.0245 & -0.0054 & -0.0129 \\ -0.0867 & -0.3109 & 0.8293 & -0.0889 & -0.0475 & -0.0816 \\ -0.0245 & -0.0381 & -0.0381 & 0.8458 & -0.0600 & -0.0418 \\ -0.0245 & -0.0631 & -0.0385 & -0.0153 & 0.9867 & -0.0150 \\ -0.0676 & -0.1106 & -0.0582 & -0.0776 & -0.1740 & 0.8240 \end{bmatrix}; \\
 &0.6633 \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} = 0.6633 \begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} +
 \end{aligned}$$

$$\begin{aligned}
 & +r0.6633 \begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} + \\
 & +w*0.6633 \begin{bmatrix} 0.9805 & -0.0025 & -0.0101 & -0.0104 & -0.0011 & -0.0012 \\ -0.0273 & 0.9992 & -0.0049 & -0.0245 & -0.0054 & -0.0129 \\ -0.0867 & -0.3109 & 0.8293 & -0.0889 & -0.0475 & -0.0816 \\ -0.0245 & -0.0381 & -0.0381 & 0.8458 & -0.0600 & -0.0418 \\ -0.0245 & -0.0631 & -0.0385 & -0.0153 & 0.9867 & -0.0150 \\ -0.0676 & -0.1106 & -0.0582 & -0.0776 & -0.1740 & 0.8240 \end{bmatrix} \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}
 \end{aligned}$$