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Pitfalls in Higher Order Model Extensions of Basic Spatial Regression Methodology

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Abstract: Spatial regression methodology has been around for most of the 50 years (1961-2011) that the Southern Regional Science Association has been in existence. Cliff and Ord (1969) devised a parsimonious specification for the structure of spatial dependence among observations that could be used to empirically model spatial interdependence. Later work (Cliff and Ord, 1973, 1981; Ord, 1975) further developed these ideas into basic spatial regression models, which were popularized and augmented by Anselin (1988). We discuss several issues that have arisen in recent work that attempts to extend basic models of spatial interdependence to include more types of spatial and non-spatial interdependencies. Understanding these issues should help future work avoid several pitfalls that plague current and past attempts at extensions along these lines.
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1. INTRODUCTION

Many of the problematical attempts to extend basic spatial regression models to include more elaborate structures of dependence result from a lack of complete understanding of basic spatial regression. Section 1.1 outlines basic spatial regression models that are in wide use, while section 1.2 discusses a frequently proposed extension of the basic model. Sections 1.3 to 1.6 set forth four pitfalls that beset the proposed extension from section 1.2 that have not been widely recognized in the literature. These appear to stem from key facets of basic spatial regression models that are misunderstood.

Section 2 turns attention to alternative approaches to extending the basic model. These include more local structures of dependence that avoid the pitfalls discussed in sections 1.3 to 1.6.

1.1. Basic spatial regression models

The most frequently used basic spatial regression models fall into two broad categories, those that model spatial dependence in the disturbances and those that treat dependence in the dependent variable. Expression (1) shows a simple specification for dependence in the

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disturbances and (2) a simple model for dependence in the dependent variable vector observations.

(1) $$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \quad \mathbf{u} = \rho \mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon}$$

(2) $$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

In both models, the $n \times 1$ vector $\mathbf{y}$ represents a dependent variable that exhibits variation across spatial observational units, and the $n \times k$ matrix $\mathbf{X}$ represents explanatory variables that usually include a vector of ones. The scalar parameter ρ measures the strength of spatial dependence with boundaries on the permissible (stationary) parameter space determined by minimum and maximum eigenvalues of the $n \times n$ matrix $\mathbf{W}$. For simplicity, we assume that $\mathbf{W}$ has all real eigenvalues and that the principal eigenvalue equals 1. The matrix $\mathbf{W}$ provides a (normalized) structure of connectivity between the observations and in spatial regression models *each observation is a region*. In a spatial context, connectivity might be defined as *neighboring regions* using non-zero elements in the i, j th position of the matrix $\mathbf{W}$ to denote that region j is a neighbor to region i. The $n \times 1$ vector $\boldsymbol{\varepsilon}$ is a disturbance term usually assumed to be normally distributed with zero mean, constant variance $\boldsymbol{\sigma}^2$ and zero covariance across observations.

The parameters of the model are $\boldsymbol{\beta}, \rho$ and $\boldsymbol{\sigma}^2$ which can be estimated using maximum likelihood, Bayesian or instrumental variable methods (see LeSage and Pace, 2009 for details).

The concept of spatial neighbors used to form the matrix $\mathbf{W}$ might be defined using: first-order contiguity (regions with borders that touch), or some number m of nearest neighboring regions based on distances. One could also define a distance cut-off of q miles and select all m_i regions within this distance to each region $i = 1, \ldots, n$ as neighboring regions/observations. It is also possible to rely on inverse distances between all regions/observations or some other function of distance with a parameter reflecting a distance decay factor. This has computational and statistical disadvantages for problems involving a large number of regions/observations, since it results in non-sparse weight matrices $\mathbf{W}$ because all elements of the matrix $\mathbf{W}$ take non-zero values. Statistically, the reliance on neighbors should not be a function of the number of observations. Put another way, in a small town it seems reasonable to assume that every property depends on every other property, but for the country as a whole it seems unappealing to say that properties in Fairbanks, Alaska materially depend on properties in Miami.

1.2. A frequently proposed extension to the basic spatial regression model

A frequently proposed extension of the basic spatial SAR model in (2) is based on adding additional connectivity matrices, which for the case of a single additional weight matrix leads to the model in (3) (see Lacombe 2004, Badinger and Egger, 2011, Elhorst, Lacombe and Piras, 2012).

(3) $$\mathbf{y} = \rho_1 \mathbf{W}_1\mathbf{y} + \rho_2 \mathbf{W}_2\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

There are two motivations given for extending the simple model from (2) using the specification in (3) . One is that additional weight matrices can capture more elaborate types of non-spatial dependence. For example, Badinger and Egger (2011) label these ``alternative modes of interdependence," and cite as an example Case et al. (1993) who use regional differences in per-capita income to form a matrix $\mathbf{W}$.

A second motivation given for extending the simple model to include additional spatial weight matrices is the *mistaken belief* that model marginal effect estimates and inferences are extremely sensitive to the spatial weight matrix used in basic spatial regression models. This belief arises since parameter estimates do change across different specifications of $\mathbf{W}$ and including additional spatial weight matrices often improves model fit. However, it is well-known in other areas of econometrics that differences in parameter estimates and model fit across models do not necessarily lead to materially different marginal effects in large samples. For example, Wooldridge (2005) shows for probit models that omitted heterogeneity (reduction of model fit) does not materially affect the average marginal effect even though it creates an attenuation bias in the parameter estimates. As another example, logit and probit models often have similar marginal effects even though their parameter estimates have different magnitudes. Therefore, it does not necessarily follow that marginal effect estimates and inferences are sensitive to small improvements in $\mathbf{W}$.

We explore these two motivations for the SAR model specification in (3) that includes an additional weight matrix in the context of four pitfalls that arise for this model specification in the next four sections.

1.3. Pitfall #1: understanding spatial spillovers

There are important distinctions between the spatial error model (SEM) in (1) and the spatial autoregressive model (SAR) in (2) , since the former does not allow for spatial spillovers. LeSage and Pace (2009) define spatial spillovers as non-zero cross-partial derivatives $\partial y_j / \partial x_i$, so that changes to explanatory variables in region i impact the dependent variable values in region $j \neq i$.

For the SEM model the cross-partial derivatives in (5) (spillovers) are zero by design, as in the case of non-spatial regression models. The SEM model estimate $\hat{\boldsymbol{\beta}}_r$ (and associated measure of dispersion) form the basis for inference regarding how changes in the rth explanatory variable in region i will impact the ith region values of the dependent variable, and this scalar estimate averages over all $i = 1, \ldots, n$ observations. In other words the SEM parameter estimate equals the average marginal effect of the own variable (average direct effect) and the average marginal effect of the spillovers is 0 (average indirect effect is 0).

(4) $$\partial y_i / \partial x_i^r = \boldsymbol{\beta}_r$$

(5) $$\partial y_j / \partial x_i^r = 0$$

In contrast to the SEM, the SAR model allows for non-zero cross-partial derivatives (spillovers) which should frequently be of interest to regional scientists. A literal interpretation of the cross-sectional SAR model would be that spillovers arise simultaneously. Since there is no explicit role for time in a cross-sectional setting, LeSage and Pace (2009) argue that spillovers in the context of spatial regression models should be interpreted as changes that will arise in the dependent variable (as a result of changes in the explanatory variables) as the relationship under study moves to a new steady-state equilibrium. Cross-sectional observations could be viewed as reflecting a slice at one point in time of a long-run steady-state equilibrium relationship, and comparative static analysis of changes then represent new steady-state relationships that would arise over time.

The ultimate goal of estimating spatial regression model parameters is inference regarding how changes in the explanatory variables impact the dependent variable which appears in (6) and (7) for the SAR models. We use $\mathbf{S}_{ij}$ in (6) and (7) to represent the i, j th element of the $n \times n$ matrix S shown in (8) .

(6) $$\partial y_i / \partial x_i^r = \mathbf{S}_{ii} \boldsymbol{\beta}_r$$

(7) $$\partial y_j / \partial x_i^r = \mathbf{S}_{ij} \boldsymbol{\beta}_r$$

(8) $$\mathbf{S} = (\mathbf{I}_n - \rho \mathbf{W})^{-1}$$

In the case of spillovers such as the SAR model, changes in the rth explanatory variable in region i will impact the ith region values of the dependent variable as shown in (6) , as well as other regions j as shown in (7) , leading to an $n \times 1$ vector of potential responses. Since we can change each of the $i = 1, \ldots, n$ regions' rth explanatory variable values, this results in an $n \times n$ matrix of own- and cross-partial derivatives. LeSage and Pace (2009) propose using the average of the main diagonal elements of the $n \times n$ matrix representing the own-partial derivatives as a scalar summary measure of *direct effects* estimates and the average of the cumulative sum of the off-diagonal elements (reflecting cross-partial derivatives) from each row as a scalar summary of *indirect effects* estimates or spillovers.

There are a great many studies employing SAR models (and other models involving spatial lags of the dependent variable, $\mathbf{Wy}$) that misinterpret the coefficient estimates for the parameters $\boldsymbol{\beta}$ as if they represent partial derivatives showing how changes in the explanatory variables impact the dependent variable. This accompanies the belief that we only need to be able to estimate the model parameters $\boldsymbol{\beta}, \rho, \boldsymbol{\sigma}^2$ and our modeling experience is complete. In this line of thinking, inference regarding the influence of the explanatory variables on the dependent variable is a trivial exercise comparable to that in non-spatial regression, since all that is needed for inference are estimates of the parameters $\boldsymbol{\beta}$ (and associated measures of dispersion). The argument is that estimation methods can produce $\boldsymbol{\beta}, \rho, \boldsymbol{\sigma}^2$ estimates for models involving more than a single weight matrix, so extensions of the basic model are quite simple. This is often cited as an area where GMM estimation holds advantages over maximum likelihood, but this ignores the fact that Maximum Likelihood and Bayesian Markov Chain Monte Carlo (MCMC) (likelihood-based methods) can easily produce estimates for the parameters $\boldsymbol{\beta}, \rho_1, \rho_2, \boldsymbol{\sigma}^2$ for models involving more than a single weight matrix.1

In fact, contrary to popular belief, drawing inferences about *direct* and *indirect* effects that can be separably attributed to more than a single type of connectivity is a major pitfall to extending basic spatial regression models in this direction. Specifically, for the specification in (3) we have an $n \times n$ matrix of partial derivatives taking the form in (9) .

[1]Pace and LeSage (2002) proposed a semiparameteric model for spatial dependence in the dependent variable that involved a large number of simpler weight matrices (termed spatial basis matrices). They illustrated their proposed method using approximate maximum likelihood estimation of a model based on a large sample of 57,647 U.S. census tracts.

(9) $$\partial y/\partial x^{r'} = (\mathbf{I}_n - \rho_1 \mathbf{W}_1 - \rho_2 \mathbf{W}_2)^{-1} \boldsymbol{\beta}_r$$

This should make it clear that the main diagonal elements (reflecting direct effects/own-partials) and off-diagonal elements (reflecting indirect effects/cross-partials) of the $n \times n$ matrix inverse involve elements of both $\mathbf{W}_1$ and $\mathbf{W}_2$ reflecting a combination of the two types of dependence being modeled. An implication is that it is not possible to easily separate out alternative transmission channels of spillovers associated with $\mathbf{W}_1$ and $\mathbf{W}_2$ types of connectivity. This should raise important concerns about the wisdom of pursuing models with a specification such as that in (3), and it suggests that portrayals of these extended models as straightforward and easy-to-understand extensions of the basic model are not accurate.

1.4. Pitfall #2: sensitivity to the weight matrix

A second motivation given for extending the simple model to include additional weight matrices is the *mistaken belief* that model marginal effect estimates and inferences are extremely sensitive to the weight matrix used in basic spatial regression models. Although including additional weight matrices may yield a closer approximation to the correct/true but unknown weight matrix, it does not necessarily follow that the marginal effects are sensitive to small changes in $\mathbf{W}$.

LeSage and Pace (2011) label the belief that spatial regression marginal effect estimates and inferences are sensitive to particular choices made regarding the matrix $\mathbf{W}$ as *the biggest myth in spatial econometrics*. They show that *contrary to popular belief*, marginal effect estimates and inferences regarding how changes in the explanatory variables in these models impact the dependent variable are not overly sensitive to alternative approaches to specifying the matrix $\mathbf{W}$ *in spatial settings*. The basic reasoning is that the correlation between various spatial lags of $\mathbf{y}$ will usually be high in a spatial setting, and therefore the additional information content from introducing additional lags is small. However, what about a non-spatial lag? Do these results still hold true? Are there additional issues that arise when using non-spatial $\mathbf{W}$?

Consider the case of $\mathbf{W}_1$ being a spatial weight matrix and $\mathbf{W}_2$ being based on some other variable vector that we label $\mathbf{z}$. Examples might be: income, age, educational attainment, employment, rental rates, or house prices. Pace, LeSage and Zhu (2011) show that the spatial dependence parameter estimates from a model: $\mathbf{z} = \boldsymbol{\alpha} + \rho \mathbf{W}\mathbf{z} + \boldsymbol{\varepsilon}$ for all of these variables ranges from a low of 0.75 to a high of 0.96, for samples of counties, census tracts and block groups. This suggests a high degree of spatial clustering in these variables, and the point made by Pace, LeSage and Zhu (2011) is that most explanatory variables used in spatial regression models exhibit a high degree of spatial dependence.

This leads to the question ─ what new information is added to the model by inclusion of *supposedly non-spatial* $\mathbf{W}_2\mathbf{y}$ as an explanatory variable? As an example, consider expressing the inverse using the series expansion:

$$\sum_{t=0}^{\infty} (\rho_1 \mathbf{W}_1 + \rho_2 \mathbf{W}_2)^t .$$

The first order ($t = 1$) term is straightforward. The second order term ($t = 2$) contains, among others, the quantities $\mathbf{W}_1^2$ and $\mathbf{W}_2^2$. For symmetric $\mathbf{W}_1$, $\mathbf{W}_2$ the diagonal elements of $\mathbf{W}_1^2$ and

$\mathbf{W}_2^2$ equal the sum of squared elements from the respective rows. Thus both $\mathbf{W}_1^2$ and $\mathbf{W}_2^2$ have large elements on the diagonals meaning these may exhibit substantial covariation, even when there is no relation between $\mathbf{W}_1$ and $\mathbf{W}_2$.

A related complication that arises from use of non-spatial matrices $\mathbf{W}_2$ is whether we can treat the matrix $\mathbf{W}_2$ as exogenous. As already noted, the own- and cross-partial derivatives used to interpret the SAR model involve the matrix $\mathbf{W}$, specifically: $\partial y/\partial x^{r'} = (\mathbf{I}_n - \rho\mathbf{W})^{-1}\boldsymbol{\beta}_r$, where only x^r is changed, not elements in the matrix $\mathbf{W}$. Of course, treating the spatial configuration of the n regions as fixed, while changing an explanatory variable x_r is conceptually realistic. Can we say the same thing about changes in explanatory variables that may be highly correlated with non-spatial variables used to construct a non-spatial connectivity matrix $\mathbf{W}_2$? For example, Kelejian and Mukerji (2011) rely on trade flows between countries as the basis for constructing a non-spatial weight matrix used in a model where one of the explanatory variables is the exchange rate. Can we truly view this type of weight matrix as exogenous in the face of changes in an explanatory variable such as the exchange rate? Consider that an important point about the structure of spatial connectivity between regions is that this tends to remain very constant over time. Can the same be said about trade flows used to construct a non-spatial weight matrix and the exchange rate?

Another example is Badinger and Egger (2010), who consider spillover effects on US foreign affiliate sales in OECD countries based on a model including two non-spatial connectivity matrices. One is constructed based on horizontal interdependence measured using bilateral final goods trade flows and the other reflecting vertical interdependence based on bilateral trade in intermediate goods. Choice of non-spatial connectivity structures such as trade or migration flows between regions that change over time would of course lead to estimates and inferences that may be sensitive to the choice of weight matrices, or estimates and inferences that vary over time.

Blankmeyer et al. (2010) address some additional issues that arise for non-spatial weight matrices in the context of constructing a set of *peer* nursing homes for a sample of around 1,000 nursing homes, all located in the state of Texas. They ignore spatial location entirely and focus on univariate or multivariate criteria of institutional similarity, such as the number of beds, size of the nursing staff, square foot area of the facility, payroll, etc. They note that conventional measures of univariate or multivariate distance (e.g. Euclidean or Mahalanobis distance) can be calculated and used to identify peer institutions to each observation as those that are most similar, that is, those that exhibit smaller distances constructed using the similarity criterion. The result is an $n \times n$ matrix of (generalized) distances between each of the n observations and all others. To define peer groups each having m institutions, they select the m nearest neighbors as the m most similar institutions.

While it is possible to calculate generalized distances between two vectors, say the number of beds and payroll, scaling becomes very important. To see this, consider that payroll might be expressed in dollar values leading to large magnitudes in the millions. This variable would have unequal weight relative to the number of beds. This is unlike the case of spatial proximity, where Euclidean distance calculated based on latitude-longitude provides a natural scaling. There is also a need to take into account covariance between the two variable vectors reflecting the number of beds and payroll of the nursing home that does not arise in the case of geographical space. The scaling problem becomes exponentially worse in the case of multivariate dimensions.

Pace, Sirmans, and Slawson (2002) studied adapting the spatial error model to reflect real estate appraisal practice. Comparable properties in real estate appraisal play the role of neighbors. Appraisal practice usually avoids using comparable properties where the number of bedrooms differs by more than 1 from the subject property (observation). Pace, Sirmans, and Slawson (2002) used this restriction, another similar restriction on bathrooms, and some restrictions based on age and housing size to estimate a multidimensional $\mathbf{W}$ beginning with a spatial $\mathbf{W}$ and zeroing out elements based on whether they violated the restrictions. Their estimate for the effects of house size was not materially different than the estimate based on a purely spatial $\mathbf{W}$. However, house size was part of $\mathbf{W}$ and also part of the model. It should be clear that this approach greatly increases the difficulty of interpreting partial derivative changes that arise from changes in house size on the dependent variable.

1.5. Pitfall #3: stationary regions for the parameter space in models involving multiple weight matrices

For the case of the basic spatial regression models in (1) and (2), it is well known that minimum and maximum eigenvalues of the weight matrix $\mathbf{W}$ determine the feasible range for the parameter ρ (see LeSage and Pace, 2009 for details). For two $\mathbf{W}$, Lee and Liu (2010) rely on the sufficient condition that the sum of the absolute values of the two spatial parameters should be less than one $(|\rho_1|+|\rho_2|<1)$. This condition can also work for additional $\mathbf{W}$ matrices. However, the more $\mathbf{W}$ matrices used, the more likely that some of the estimated dependence parameters will lie on or very near the boundary of the parameter space. However, using multiple $\mathbf{W}$ matrices creates some computational and inferential issues since the parameter space is restricted and estimation methods such as maximum likelihood and GMM require estimates to lie sufficiently within the interior of the parameter space to carry out conventional inference.[2] To the degree that the estimation of the individual ρ parameters becomes less precise, this imprecision means that the parameter estimates could fall outside the feasible region in repeated sampling and thus invalidate the asymptotics used in conventional inference. In other words, using more $\mathbf{W}$ matrices increases the number of boundaries and may increase the imprecision of estimation which reduces the chance that the parameter estimates will fall in the interior of the boundaries in repeated sampling, a requirement underlying conventional inference.

The simplest approach to this problem would be to impose such restrictions during Bayesian MCMC estimation where samples are drawn (using a Metropolis-Hastings (M-H) procedure) from the conditional distribution for parameter ρ_1 given all other parameters (including ρ_2), and samples from the conditional distribution for parameter ρ_2 given all other parameters (including ρ_1). The feasible range for ρ_1 *given* ρ_2 is easy to calculate, and M-H candidate draws that lie outside the feasible range can simply be rejected, a method known as

[2]One could perhaps reduce this problem by going to the more permissive regions proposed by Elhorst, Lacombe and Piras (2012). However, this more permissive region does not cure the possible parameter estimate on the boundary problem and this approach becomes very complicated when using multiple $\mathbf{W}$ matrices.

rejection sampling. A similar statement applies to the M-H process for constructing draws for the parameter ρ_2 conditional on (given) ρ_1 values. Note, the Bayesian MCMC approach would allow for correct inference of the marginal effects even in the presence of these restrictions.

1.6. Pitfall #4: the correct specification for models involving multiple weight matrices

The model specification in (3) seems to be widely viewed as a *natural* extension of the basic spatial lag (SAR) model, but there are other competing specifications that could be employed. One alternative is the specification in (10), which can be viewed as *filtering* the dependent variable vector $\mathbf{y}$ for two types of dependence, one involving $\mathbf{W}_1$ and the other $\mathbf{W}_2$. This leads to the model statement in (11), where we see a matrix product $\mathbf{W_1 W_2}$ involving the two types of dependence that is a logical result of this type of specification.

(10) $$(\mathbf{I_n} - \rho_1 \mathbf{W_1})(\mathbf{I_n} - \rho_2 \mathbf{W_2})\mathbf{y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$$

(11) $$\mathbf{y} = \rho_1 \mathbf{W_1 y} + \rho_2 \mathbf{W_2 y} - \rho_1 \rho_2 \mathbf{W_1 W_2 y} + \mathbf{X\beta} + \boldsymbol{\varepsilon}$$

(12) $$\mathbf{y} = \mathbf{S^{-1} X\beta} + \mathbf{S^{-1}}\boldsymbol{\varepsilon}$$
$$\mathbf{S} = [(\mathbf{I_n} - \rho_1 \mathbf{W_1})(\mathbf{I_n} - \rho_2 \mathbf{W_2})]$$

For ease of exposition, let us denote **A**= $(\mathbf{I_n} - \rho_1 \mathbf{W_1})$ and **B**= $(\mathbf{I_n} - \rho_2 \mathbf{W_2})$ and $\mathbf{S} = \mathbf{AB}$. The reduced form of the model is shown in (12), where we note the matrix algebra rules: $\mathbf{(AB)^{-1}} = \mathbf{B^{-1}A^{-1}}$ and $\mathbf{A^{-1}} = \mathbf{I_n} + \rho_1 \mathbf{W_1} + \rho_1^2 \mathbf{W_1^2} + \ldots$, while $\mathbf{B^{-1}} = \mathbf{I_n} + \rho_2 \mathbf{W_2} + \rho_2^2 \mathbf{W_2^2} + \ldots$, where the expressions for $\mathbf{A^{-1}}, \mathbf{B^{-1}}$ are infinite series. This suggests that numerous cross-product terms would arise involving $\mathbf{W_1 W_2}, \mathbf{W_1 W_2^2}, \ldots$ and $\mathbf{W_2 W_1}, \mathbf{W_2 W_1^2}, \ldots$ in the reduced form inverse expression. We also note that in general $\mathbf{W_1 W_2} \neq \mathbf{W_2 W_1}$.

How does this specification compare to that in (3) ? One can view the specification in (3) as a *special case* of this specification that arises only when the matrix products: $\mathbf{W_1 W_2} = \mathbf{0}_{n \times n}$, and $\mathbf{W_2 W_1} = \mathbf{0}_{n \times n}$, *as well as* all other matrix cross-products involving higher-order powers of $\mathbf{W_1^s}, \mathbf{W_2^t} = 0, s = 1, t = 1, t = 2, \ldots, s = 2, t = 1, t = 2, \ldots$, etc.

An important point to note is that the *order* in which we enter the matrices $\mathbf{W_1}, \mathbf{W_2}$ in the model matters, since in general $\mathbf{W_1 W_2} \neq \mathbf{W_2 W_1}$. This means that changing the order in which we *filter* for the two types of dependence embodied in $\mathbf{W_1}, \mathbf{W_2}$ would lead to different coefficient estimates. Again, the upshot of this is that the model specification in (3) has another implicit assumption that the matrix products: $\mathbf{W_1 W_2} = \mathbf{W_2 W_1}$, as does the more general specification in (10). It should be clear to see that extension to cases involving *more than two* $\mathbf{W}$*-matrices* would rapidly become complicated.

Another implication of these results is that interpretation of separate spillover impacts from the two types of connectivity embodied in the more general models would be very difficult. These are shown in (13) using the $n \times n$ matrix of own- and cross-partial derivatives that would be used to calculate direct and indirect effects.

(13) $$\partial y / \partial x^{r'} = \mathbf{S}^{-1} \boldsymbol{\beta}_r$$

$$\mathbf{S}^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$$
(14) $$= (\mathbf{I}_n + \rho_2 \mathbf{W}_2 + \rho_2^2 \mathbf{W}_2^2 + \ldots)(\mathbf{I}_n + \rho_1 \mathbf{W}_1 + \rho_1^2 \mathbf{W}_1^2 + \ldots)$$

For the restrictive case where the order of $\mathbf{W}_1$ and $\mathbf{W}_2$ were known apriori, one could interpret these effects estimates in the following way.[3] Assuming that $\mathbf{W}_2$ represents a spatial weight matrix and $\mathbf{W}_1$ technological connectivity between regions, the total effects of technological connectivity (direct plus spillovers) arising from a change in the rth variable are embodied in the matrix $\mathbf{A}^{-1}\boldsymbol{\beta}_r$. The global spatial spillovers matrix $\mathbf{B}^{-1}$ then aggregates total technological effects falling on spatially nearby regions ($\mathbf{W}$), neighbors to these regions ($\mathbf{W}^2$), neighbors to the neighbors ($\mathbf{W}^3$) and so on (see LeSage and Fischer, 2012).

1.7. Conclusions regarding simple extensions of the basic spatial model

The belief expressed in the literature that extending the basic single spatial weight matrix model using the specification in (3) is relatively straightforward is naive. It overlooks four important pitfalls that arise. The pitfall related to technical issues regarding the feasible parameter space could be resolved using Bayesian MCMC estimation of the model, but those pertaining to interpretation of *separable* spillovers for additional types of connectivity introduced using the specification in (3) are more challenging.

Two motivations for extension of the basic SAR spatial regression model along the lines of (3) given in the literature seem to be naïve as well. One motivation is based on the *mistaken* belief that marginal effect estimates and inferences are sensitive to use of the correct spatial weight matrix, and the other assumes that spillovers associated with additional connectivity structures introduced using the specification (3) can be *separately* analyzed.

Issues relating to what is the appropriate specification to use in extending the basic SAR spatial regression raised in section 1.6 may explain why the complicated feasible parameter space arises for the specification in (3) . If this specification imposes some arbitrary restrictions regarding covariances (cross-products) between multiple types of connectivity structures, these restrictions could account for the non-linear relationship between feasible values for the dependence parameters.

Finally, the desire to consider non-spatial types of connectivity must be tempered by a realization that a host of technical issues arise with regard to how one proceeds to specify non-spatial connectivity relationships. Spatial connectivity relations specified in the way set forth by Cliff and Ord (1969) hold a great many of advantages that do not carry over to generalized measures of distance between regions.

2. APPROACHES THAT AVOID THE PITFALLS

One misunderstanding about basic spatial regression models appears to be a lack of understanding about the distinction between *local* versus *global* spillovers (Anselin, 2003). An

[3]Recall that order of the matrices $\mathbf{W}_1, \mathbf{W}_2$ makes a difference in the magnitude of the effects estimates.

attractive motivation for use of spatial regression models by regional scientists is their ability to provide quantitative estimates of the magnitude of these two types of spillovers.

Global spillovers are those arising from spatial lags that lead to reduced form expressions involving: $(\mathbf{I_n} - \rho\mathbf{W})^{-1} = \mathbf{I_n} + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \ldots$, which allow for spillovers to neighbors, neighbors to neighbors, and so on, which can emanate out to very high-order neighbors (more distant regions). In contrast, local spillovers are those associated with immediate neighbors that do not exert influence on higher-order neighboring observations.

This section discusses model specifications that produce *local* spatial spillovers associated with explanatory variables in the model that are *separable*, which have been generally overlooked in the spatial econometrics literature. It is argued that in many applications interest should perhaps focus on how *local* spillovers arise from changes in the explanatory variables.

In terms of local models, LeSage and Pace (2009) examine spatially lagged explanatory variables (SLX) and spatial Durbin error models (SDEM). These models might avoid pitfalls that arise with naïve approaches to extending models to include more than a single weight matrix. The SLX model is shown in (15) and the SDEM model in (16), where spatial lags of the explanatory variables are included in both models, but no spatial lag of the dependent variable.

(15) $$\mathbf{y} = \mathbf{X\beta} + \mathbf{WX\theta} + \boldsymbol{\varepsilon}$$

(16) $$\mathbf{y} = \mathbf{X\beta} + \mathbf{WX\theta} + \mathbf{u}$$
$$\mathbf{u} = \rho\mathbf{Vu} + \boldsymbol{\varepsilon}$$

The SLX model allows for *local* spatial spillovers which can be directly calculated using the coefficients $\boldsymbol{\theta}$ since:

(17) $$\partial y/\partial x^{r'} = (\mathbf{I_n\beta_r} + \mathbf{W\theta}_r)$$

The average of the main diagonal elements of the $n \times n$ matrix $(\mathbf{I_n\beta_r} + \mathbf{W\theta_r})$ reflects the direct effects, while the cumulative indirect effects can be constructed using an average of the (cumulated) off-diagonal elements. Further noting that the main diagonal of the matrix $\mathbf{W}$ contains zeros, (so regions cannot be neighbors to themselves) and the rows of the matrix $\mathbf{W}$ sum to one, leads to the simple conclusion that the coefficient $\boldsymbol{\beta}_\mathbf{r}$ reflects direct effects while $\boldsymbol{\theta}_\mathbf{r}$ captures spatial spillovers. The SDEM model allows for these same local spillovers with regard to the explanatory variables, but also models global spatial dependence in the disturbance structure of the model, using the spatial autoregressive process $\mathbf{u} = \rho\mathbf{Wu} + \boldsymbol{\varepsilon}$.

For the SDEM and SLX models, the coefficients in the vector $\boldsymbol{\theta}$ represent spillovers that impact the immediately neighboring observations. We note that estimates from these two models should be similar, but in the face of spatial dependence in the disturbances, SDEM model estimates should be more efficient. The partial derivative expressions for both models are the same, but improved efficiency for the case of the SDEM model could impact inferences regarding significance of the direct and indirect effects estimates. Pace and LeSage (2008) provide a Hausman test that could be used to test for equality of the SLX and SDEM model coefficients. An absence of equality for these two sets of coefficients may provide evidence against the SDEM model in favor of a spatial lag variant such as the SAR or SDM models (see LeSage and Pace, 2009).

In many cases local spillovers are actually the focus of interest, a point that has often been misunderstood in the spatial econometrics literature. For example, Lacombe (2004) uses the model specification in (3), where $\mathbf{W}_1$ represents neighboring counties within the state and $\mathbf{W}_2$ is constructed using neighboring counties across the state border. His sample data consists only of border counties and interest centers on the effects of state-level variation in Aid to Families with Dependent Children (AFDC) and Food Stamp program benefit levels on female labor force participation. Changes in benefit levels of these two state administered aid programs for low-income residents could have a spillover impact on counties in neighboring states, since it is possible for residents of border counties to simply move to neighboring states if there is a large discrepancy in aid benefits between neighboring states. This would seem to be a case where local spillovers should be the focus of interest. Use of the variant of the SLX model in (18) would allow interpreting the coefficient θ_r associated with say AFDC benefit levels in the model, as a measure of how changes in state-level AFDC benefits (on average over the sample of border counties) impact own-state female labor market participation (direct effects). This is because there is no difference in own-county and own-state benefit levels associated with these state administered programs.

(18) $$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}_1\mathbf{X}\boldsymbol{\theta} + \mathbf{W}_2\mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$$

The estimate $\boldsymbol{\gamma}_r$ would measure the impact of changes in AFDC benefit levels in neighboring states on border counties female labor force participation, reflecting the spillover impact from changes in the neighboring state AFDC benefit levels on labor force participation, again averaged over all border counties in the sample. An important point is that the SLX/SDEM model specification does allow *separation* of the two types of spillover impacts, which we showed was very difficult for the model specification in (3) .

Would we really expect that changes in state-level AFDC and Food Stamp aid would lead to global spillovers? If so, changes in aid levels in Ohio could impact labor force participation in states neighboring Ohio, neighbors to the Ohio neighbors, neighbors to those neighbors, and so on. The implication is that changes in aid levels in Ohio would exert an impact on female labor market participation in states as distant as Maine and California.

An example that uses the SDEM model is LeSage and Ha (2012), who study the impact of migration on county-level social capital. Their SDEM model takes the form in (19) , where $\mathbf{W}_\mathbf{n}$ and $\mathbf{W}_\mathbf{f}$ represent migration-weighted spatial weight matrices. The matrix $\mathbf{W}_\mathbf{n}$ identifies neighboring counties within 40 miles and assigns relative weights to these based on in-migration magnitudes. The matrix $\mathbf{W}_\mathbf{f}$ identifies neighboring counties more than 40 miles away from each county that provide in-migration to each county i in the sample, and weights these according to in-migration magnitudes. The matrix $\mathbf{V}$ used to model dependence in the model disturbances was a spatial contiguity weight matrix, with equal weights assigned to all contiguous counties.

(19) $$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}_\mathbf{n}\mathbf{X}\boldsymbol{\theta} + \mathbf{W}_\mathbf{f}\mathbf{X}\boldsymbol{\gamma} + \mathbf{u}, \quad \mathbf{u} = \rho\mathbf{V}\mathbf{u} + \boldsymbol{\varepsilon}$$

Pace and Zhu (2012) point out that a desirable aspect of the model in (19) is that dependence in the disturbances is modeled separately from spillovers, which is not the case for the extended variant of the SAR model. For the basic spatial SAR model, the dependence structure for the disturbances is restricted to be the same as that for the *mean* model, which can be seen from: $\mathbf{y} = (\mathbf{I_n} - \rho\mathbf{W})^{-1}\mathbf{X\beta} + (\mathbf{I_n} - \rho\mathbf{W})^{-1}\boldsymbol{\varepsilon}$. This implies that the expectation for $\mathbf{y}$ is a function of ρ and $\mathbf{W}$, $\mathbf{E(y)} = (\mathbf{I_n} - \rho\mathbf{W})^{-1}\mathbf{X\beta}$, and the disturbance covariance $\boldsymbol{\Omega} = \boldsymbol{\sigma}^2[(\mathbf{I_n} - \rho\mathbf{W})^{-1}(\mathbf{I_n} - \rho\mathbf{W'})^{-1}]$ takes the same functional form. Of course, the specification in (3) is even more restrictive in this regard, forcing the mean part of the model and disturbance covariance to take a more restrictive form involving two weight matrices and two spatial dependence parameters. Misspecification in one part of the model will then contaminate the other aspect of the model specification.

The SDEM model in (19) allows *separation* of the (local) spillover impacts on county-level social capital levels arising from changes in population characteristics of nearby counties (providing in-migrants to each county in the sample) versus that arising from changes in population characteristics of far away (outside the region) counties (providing in-migrants to each county). For example, how do changes in educational attainment levels of population in counties within the region versus counties outside the region (providing in-migrants to each county) impact levels of social capital in the typical county? Are there important differences in the magnitude of impact associated with in-migration from within and outside the region? Are some changes in characteristics of in-migrants from nearby counties significant/insignificant while the same characteristics of in-migrants from outside the region are insignificant/significant?

By way of conclusion, we note that although the term *local* spillovers could be used to characterize the model in (19), this does not necessarily rule out consideration of spillover impacts involving great distances, since practitioners can try variants of individual $\mathbf{W}$ with different bandwidths to capture longer range dependencies. Therefore, practitioners of spatial regression models should spend time thinking about whether the phenomena being modeled are likely to produce local or global spillovers.

3. CONCLUSION

We point out four pitfalls associated with a popular but perhaps naïve extension of the basic single weight matrix spatal model specification to the case of more than one weight matrix, which has been frequently utilized in the spatial econometrics literature. These pitfalls appear to arise from some fundamental misconceptions regarding estimation and interpretation of the basic SAR spatial regression model. We also argue that two motivations given for the extended spatial regression appear to arise from an equally simplistic view of spatial regression models.

Extending single weight matrix models to include more than spatial connectivity relationships opens up a host of issues, some pertaining to estimation, and many more to how we interpret resulting estimates from these extended models to draw inferences. Past applications of models containing multiple weight matrices that can be found in the literature have not generally given careful consideration to these issues.

We raise some questions about whether the motivations given for extending basic spatial models to include more than a single weight matrix are well thought-out, or if they perhaps arise from confusion regarding how we should model spillovers. If substantive interest in spillovers requires a separation of the magnitude and channels of impact associated with multiple types of connectivity between regions, only model specifications that focus on local spillovers allow clear-cut separation. Global spillovers are almost by definition not separable, since higher-order interactions in geographical space lead to an overlap in nearby and distant spillovers. When considering spatial and non-spatial types of connectivity between regions, additional issues arise regarding covariance between these different types of connectivity, that have for the most part been ignored in the applied literature.

Past work in spatial econometrics has lead to a great deal of progress regarding estimation of the parameters in a host of different types of useful spatial econometric model relationships. These include models for: spatially dependent origin-destination flows, space-time panel relationships, spatial Tobit and probit relations and models with endogenous explanatory variables. Future work in spatial econometrics needs to turn attention to how these estimates can be used to draw proper inferences regarding the relationships we are modeling. This should, after all, be the ultimate goal of estimating regional science relationships using spatial econometric models. The thesis of this article is that past work on extending basic spatial regression models has placed too much emphasis on estimation of the parameters while ignoring how to properly view the resulting estimates.

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